1 - Number bases

The **natural numbers** are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ...

often written in base 10 or decimal which uses 10 symbols:

0 , 1 , 2 , ... , 9 (our number system)
circle line swan

Base 2 or binary uses 2 symbols:

0 , 1 (computer)

Base 3 or ternary uses 3 symbols:

0, 1, 2 (alternate computing)

Base 16 or hexadecimal uses 16 symbols:

0, 1, 2, ..., 9, A, B, C, D, E, F
A = 10, B = 11, C = 12, D = 13, E = 14, F = 15

It helps chunk 4 digits of binaries together for human-readability:

$\underbrace{1001}\underbrace{1110}_2 = 9E_{16}$

Decimal (Base 10)	Binary (Base 2)	Ternary (Base 3)	Hexadecimal (Base 16)
0	0	0	0
1	1	1	1
2	10	2	2
3	11	10	3
4	100	11	4
5	101	12	5
6	110	20	6
7	111	21	7
8	1000	22	8
9	1001	100	9
10	1010	101	A
11	1011	102	B
12	1100	110	C
13	1101	111	D
14	1110	112	E
15	1111	120	F
16	10000	121	10
17	10001	122	11
18	10010	200	12
19	10011	201	13
20	10100	202	14

Base 60? Babylonian, time.
Base 20? "Four score and seven years ago"

$10^2 = 10 \cdot 10$ $10^0 = 1$

Conversion to decimals

- ~~Fa~~ Decimal: $125 = 100 + 20 + 5 = 1 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0$
- ~~Ba~~ Binary: $1101_2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 8 + 4 + 1 = \boxed{13}$
- Hexadecimal: $1\underset{11}{b}\underset{14}{e}_{16} = 1 \cdot 16^2 + 11 \cdot 16^1 + 14 = 256 + 176 + 14 = 446$
- In general, a base b number looks like

$$(a_n a_{n-1} \cdots a_1 a_0)_b = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0,$$

where $a_i \in \{0, 1, 2, \ldots, b-1\}$.

Conversion from decimals

we can go backwards via repeated division:

Division algorithm: If a, b are natural numbers, then there exist unique natural numbers q and r such that $b = qa + r$ with $0 \le r < a$.

"quotient" (q) "remainder" (r)

Ex 1

$$16 \overline{)193} = 12 \text{ R}1 \qquad \begin{array}{r} 193 \\ -16 \\ \hline 33 \\ -32 \\ \hline 1 \end{array}$$

$\Rightarrow \underline{193} = 12 \cdot \underline{16} + 1.$

Ex 2 Convert 21 to ternary.

A

$$\begin{aligned} 21 &= 7 \cdot 3 + 0 \\ 7 &= 2 \cdot 3 + 1 \\ 2 &= 0 \cdot 3 + 2 \end{aligned}$$

read up $\Rightarrow 21 = \boxed{210_3}$.

Check answer: $210_3 = 2 \cdot 9 + 1 \cdot 3 + 0 = 18 + 3 = 21$, ✓.

Why work? Substitute back:

$$21 = (2 \cdot 3 + 1) \cdot 3 + 0 = ((0 \cdot 3 + 2) \cdot 3 + 1) \cdot 3 + 0 = 0 \cdot 3^3 + 2 \cdot 3^2 + 1 \cdot 3 + 0 = 210_3.$$

Moral: Repeated divisions by 3 extract higher coefficients of 3.

Ex 3 Convert 100110_2 to ternary.

A $100110_2 = 2^5 + 2^2 + 2 = 32 + 4 + 2 = 38.$

$$\begin{aligned} 38 &= 12 \cdot 3 + \underline{2} \\ 12 &= 4 \cdot 3 + \underline{0} \\ 4 &= 1 \cdot 3 + \underline{1} \\ 1 &= 0 \cdot 3 + \underline{1} \end{aligned}$$

$\Rightarrow \boxed{100110_2 = 1102_3}$

~~Ex A~~ ~~The~~ RGB (red-green-blue) color ~~space is~~ is a tuple (r, g, b) where $0 \le r, g, b \le 255$: $r = 0$ means red light is off, $r = 255$ means ~~red~~ light is fully on, so

$(255, 0, 0) = $ red, $(0, 255, 0) = $ green, $(0, 0, 0) = $ black, $(255, 255, 255) = $ white.

Ex 4 Convert RGB (255, 193, 204) to HEX color code (base 16).

A
- $255 = 256 - 1 = 16^2 - 1 = 100_{16} - 1 = ff_{16}.$
- $193 \underset{\text{ex1}}{=} \underline{12} \cdot 16 + \underline{1} = c1_{16}$
- $204 = \underline{12} \cdot 16 + \underline{12} = cc_{16}$

+11

So in base 16, (ff_{16}; $c1_{16}$, cc_{16}) or more commonly as the HEX code $\boxed{\#\,\underline{ff}\,\underline{c1}\,\underline{cc}}$ bubblegum pink.

In-class exercises

1. Convert the hexadecimal number ABC_{16} to decimal and to binary.
2. Convert $1D_{16}$ to ternary.
3. Compute the sum $17_{16} + AB_{16}$ and write the answer as a hexadecimal.

Solutions

(1) $ABC_{16} = \boxed{\underline{1010}\ \underline{1011}\ \underline{1100}_2}$

$ABC_{16} = 10 \cdot 16^2 + 11 \cdot 16 + 12 = 2560 + 176 + 12 = \boxed{2748}$

(2) $1D_{16} = 16 + 13 = 29 = 27 + 2 = 3^3 + 2 = \boxed{1002_3}$

(3) We can first convert to base 10, do the addition, then convert the result back to base 16, so we can do the addition in base 16 noting that $7 + B = 7 + 11 = 18 = \underline{12}_{16}$ requires a carrying:

$$\boxed{\begin{array}{r} \overset{1}{1}7_{16} \\ +\ AB_{16} \\ \hline 2_{16} \end{array}} \Longrightarrow \begin{array}{r} \overset{1}{1}7_{16} \\ +\ AB_{16} \\ \hline C2_{16} \end{array}$$

$\boxed{1+1+A=C}$

Sol 2 $17_{16} + AB_{16} = 23 + 171 = 194 = \underline{C2}_{16}$.